

# PRINCIPAL FUNCTIONS FOR BI-FREE CENTRAL LIMIT DISTRIBUTIONS

KENNETH J. DYKEMA AND WONHEE NA

**ABSTRACT.** We find the principal function of the completely non-normal operator  $l(v_1)+l(v_1)^*+i(r(v_2)+r(v_2)^*)$  on a subspace of the full Fock space  $\mathcal{F}(\mathcal{H})$  which arises from a bi-free central limit distribution. As an application, we find the essential spectrum of this operator.

## 1. INTRODUCTION AND PRELIMINARIES

Bi-free independence was introduced by Voiculescu as a generalization of free independence in a non-commutative probability space  $(\mathcal{A}, \varphi)$ . He considered a two-faced family of non-commutative random variables,  $(X_1, X_2)$ , in  $(\mathcal{A}, \varphi)$  and the expectation values for such a combined system of left and right variables. In [7], Voiculescu proved a bi-free central limit theorem and described the family of distributions that appear as limits. These are called bi-free central limit distributions.

**1.1. Bi-freeness.** Let  $z = ((z_i)_{i \in I}, (z_j)_{j \in J})$  be a two-faced family in a non-commutative probability space  $(\mathcal{A}, \varphi)$  where  $I$  and  $J$  are disjoint index sets.

**Definition 1.1** ([7]). The two-faced families  $z'$  and  $z''$  are said to be *bi-freely independent* (abbreviated *bi-free*) if there exist two vector spaces  $(\mathcal{X}', \mathcal{X}'^\circ, \xi')$  and  $(\mathcal{X}'', \mathcal{X}''^\circ, \xi'')$  with specified state vectors and unital homomorphisms  $l^\epsilon : \mathbb{C}\langle z_i^\epsilon | i \in I \rangle \rightarrow \mathcal{L}(\mathcal{X}^\epsilon)$  and  $r^\epsilon : \mathbb{C}\langle z_j^\epsilon | j \in J \rangle \rightarrow \mathcal{L}(\mathcal{X}^\epsilon)$  with  $\epsilon \in \{', ''\}$  such that the two-faced families  $T^\epsilon = ((\lambda^\epsilon \circ l^\epsilon(z_i^\epsilon))_{i \in I}, (\rho^\epsilon \circ r^\epsilon(z_j^\epsilon))_{j \in J})$  have the same joint distribution in  $(\mathcal{L}(\mathcal{X}), \varphi_\xi)$  as  $z'$  and  $z''$  where  $(\mathcal{X}, \mathcal{X}^\circ, \xi) = (\mathcal{X}', \mathcal{X}'^\circ, \xi') * (\mathcal{X}'', \mathcal{X}''^\circ, \xi'')$  and  $\lambda^\epsilon$  and  $\rho^\epsilon$  are left and right representations of  $\mathcal{L}(\mathcal{X}^\epsilon)$  on  $\mathcal{L}(\mathcal{X})$ .

**Definition 1.2** ([7]). For each map  $\alpha : \{1, \dots, n\} \rightarrow I \amalg J$  there is a unique universal polynomial  $R_\alpha$  in commuting variables  $X_{\alpha(k_1) \dots \alpha(k_r)}$ ,  $1 \leq k_1 < \dots < k_r \leq n$  such that

- (i)  $R_\alpha$  is homogeneous of degree  $n$  where  $X_{\alpha(k_1) \dots \alpha(k_r)}$  is assigned degree  $r$ ,
- (ii) the coefficient of  $X_{\alpha(1) \dots \alpha(n)}$  is 1, and
- (iii) if  $z' = ((z'_i)_{i \in I}, (z'_j)_{j \in J})$  and  $z'' = ((z''_i)_{i \in I}, (z''_j)_{j \in J})$  are bi-free pairs of two-faced families of non-commutative random variables in  $(\mathcal{A}, \varphi)$ , then

$$R_\alpha(z') + R_\alpha(z'') = R_\alpha(z' + z'')$$

where  $R_\alpha(z) = R_\alpha(\varphi(z_{\alpha(k_1)}) \dots z_{\alpha(k_r)}) | 1 \leq k_1 < \dots < k_r \leq n$ .

These polynomials  $R_\alpha$  are called *bi-free cumulants*.

**Theorem 1.3** ([7]). A two-faced family  $z$  of non-commutative random variables has a bi-free central limit distribution if and only if  $R_\alpha(z) = 0$  whenever  $\alpha : \{1, \dots, n\} \rightarrow I \amalg J$  with  $n = 1$  or  $n \geq 3$ .

We now recall the notion of a two-faced system with rank  $\leq 1$  commutation given in [8].

**Definition 1.4.** An *implemented non-commutative probability space* is a triple  $(\mathcal{A}, \varphi, P)$  where  $(\mathcal{A}, \varphi)$  is a non-commutative probability space and  $P = P^2 \in \mathcal{A}$  is an idempotent so that

$$PaP = \varphi(a)P \text{ for all } a \in \mathcal{A}.$$

*Date:* October 12, 2015.

*2000 Mathematics Subject Classification.* 46L54 (47A65).

*Key words and phrases.* Bi-freeness, bi-free central limit distribution, principal function.

Research supported in part by NSF grant DMS-1202660.

An *implemented  $C^*$ -probability space*  $(\mathcal{A}, \varphi, P)$  will satisfy additional requirements that  $(\mathcal{A}, \varphi)$  is a  $C^*$ -probability space and that  $P = P^*$ . If a two-faced family  $((z_i)_{i \in I}, (z_j)_{j \in J})$  in an implemented non-commutative probability space  $(\mathcal{A}, \varphi, P)$  satisfies that

$$[z_i, z_j] = \lambda_{i,j} P \text{ for some } \lambda_{i,j} \in \mathbb{C}, i \in I, j \in J,$$

then the family  $((z_i)_{i \in I}, (z_j)_{j \in J})$  is called a *system with rank  $\leq 1$  commutation* where  $(\lambda_{i,j})_{i \in I, j \in J}$  is the coefficient matrix of the system.

**Definition 1.5.** Let  $\mathcal{H}$  be a complex Hilbert space. Then the *full Fock space* on  $\mathcal{H}$  is

$$\mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes n}$$

where  $\Omega$  is called the *vacuum vector* and has norm one. The *vacuum expectation* is defined as  $\varphi_\Omega = \langle \cdot, \Omega \rangle$  on  $\mathcal{F}(\mathcal{H})$ . For  $\xi \in \mathcal{H}$ , the *left creation operator*  $l(\xi) \in B(\mathcal{F}(\mathcal{H}))$  is given by the formulas  $l(\xi)\Omega = \xi$  and

$$l(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n$$

for all  $n \geq 1$  and  $\xi_1, \dots, \xi_n \in \mathcal{H}$ . The adjoint  $l(\xi)^*$  of  $l(\xi)$  is called the *left annihilation operator*. The *right creation operator*  $r(\xi) \in B(\mathcal{F}(\mathcal{H}))$  is determined by the formulas  $r(\xi)\Omega = \xi$  and

$$r(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi$$

for all  $n \geq 1$  and  $\xi_1, \dots, \xi_n \in \mathcal{H}$ . The adjoint  $r(\xi)^*$  of  $r(\xi)$  is called the *right annihilation operator*.

**Theorem 1.6** (Theorem 7.4 of [7]). *For each matrix  $C = (C_{kl})_{k,l \in I \amalg J}$  with complex entries, there is exactly one bi-free central limit distribution  $\varphi_C : \mathbb{C}\langle Z_k | k \in I \amalg J \rangle \rightarrow \mathbb{C}$  so that*

$$\varphi_C(Z_k Z_l) = C_{kl} \text{ for each } k, l \in I \amalg J.$$

*If  $h, h' : I \amalg J \rightarrow \mathcal{H}$  are maps into the Hilbert space  $\mathcal{H}$  and we define*

$$z_i = l(h(i)) + l^*(h'(i)) \text{ if } i \in I$$

$$z_j = r(h(j)) + r^*(h'(j)) \text{ if } j \in J$$

*then  $z = ((z_i)_{i \in I}, (z_j)_{j \in J})$  has a bi-free central limit distribution  $\varphi_C$  where  $C_{kl} = \langle h(l), h'(k) \rangle$ . Every bi-free central limit distribution when  $I$  and  $J$  are finite can be obtained in this way.*

**Remark 1.7.** The bi-free two-faced system in Theorem 1.6 is an example of rank  $\leq 1$  commutation. Indeed,  $(B(\mathcal{F}(\mathcal{H})), \varphi_\Omega, P)$  is an implemented  $C^*$ -probability space where  $\varphi_\Omega$  is the vacuum expectation and  $P$  is a projection on  $\mathbb{C}\Omega$ . We have  $[z_i, z_j] = (\langle h(j), h'(i) \rangle - \langle h(i), h'(j) \rangle)P$ .

**1.2. Principal function of a completely non-normal operator.** Let  $T$  be a completely non-normal operator on a Hilbert space  $\mathcal{H}$  with self-commutator  $T^*T - TT^* = -2C$  that is trace class. Set  $U = \frac{1}{2}(T + T^*)$  and  $V = -\frac{1}{2}i(T - T^*)$ . Consider the  $C^*$ -algebra generated by  $C$  and the identity operator on  $\mathcal{H}$ ; it is isometrically isomorphic to  $C(\sigma(C))$ , the complex valued continuous functions on  $\sigma(C)$ , by the Gelfand-Naimark theorem. Consider the function on  $\sigma(C)$ ,

$$t \mapsto \begin{cases} -i\sqrt{-t}, & t < 0 \\ 0, & t = 0 \\ \sqrt{t}, & t > 0 \end{cases}$$

and there exists the unique element  $\hat{C}$  in the  $C^*$ -algebra corresponding to this function by the Gelfand transform. Note that  $\hat{C}^2 = C$  and  $\hat{C}\hat{C}^* = \hat{C}^*\hat{C} = |C|$ .

The *determining function* of the operator  $T$  is defined to be

$$E(l, s) = I + \frac{1}{i}\hat{C}(V - l)^{-1}(U - s)^{-1}\hat{C}$$

for  $l \in \mathbb{C} \setminus \sigma(V)$  and  $s \in \mathbb{C} \setminus \sigma(U)$ . Then  $E(l, s)$ , for each fixed  $l$  and  $s$ , is an invertible element in the  $C^*$ -algebra generated by  $T$  and  $I$ . Since  $\det(I + AB) = \det(I + BA)$  when  $A$  is compact with  $AB$  and  $BA$  in trace class, we have

$$\begin{aligned} \det E(l, s) &= \det \left( I + \frac{1}{i} C(V - l)^{-1}(U - s)^{-1} \right) \\ &= \det \left( (V - l)(U - s)(V - l)^{-1}(U - s)^{-1} \right). \end{aligned} \quad (1)$$

The *principal function*  $g$  is defined in [4] to be the element of  $L_1(\mathbb{R}^2)$  such that

$$\det E(l, s) = \exp \left( \frac{1}{2\pi i} \iint g(\delta, \gamma) \frac{d\delta}{\delta - l} \frac{d\gamma}{\gamma - s} \right). \quad (2)$$

It is known that  $\text{supp}(g)$  is contained in  $\{(\delta, \gamma) \in \mathbb{R}^2 \mid \gamma + i\delta \in \sigma(T)\}$ . Moreover, it is a complete unitary invariant for  $T$  if  $C$  has one dimensional range; that is, two completely non-normal operators  $T$  and  $T'$  are unitarily equivalent if and only if their principal functions agree, assuming each of  $T$  and  $T'$  has a self-commutator with one dimensional range. In Theorem 8.1 of [4], it is proved that

$$g(\delta, \gamma) = \text{ind}(T - (\gamma + i\delta))$$

if  $\gamma + i\delta$  is not in the essential spectrum  $\sigma_e(T)$ . This result implies that the principal function  $g$  of  $T$  is an extension of the Fredholm index of  $T - z$  to the whole plane. However, it is not the typical situation that  $g$  assumes only integer values on the plane; indeed the map  $T \mapsto g$  is onto, namely (see [5]), any summable function on  $\mathbb{R}^2$  with compact support is the principal function of a completely non-normal operator with a trace class self-commutator.

## 2. THE PRINCIPAL FUNCTION OF CERTAIN OPERATORS

2.1. Let  $\mathcal{H}$  be a Hilbert space and  $v_1, v_2 \in \mathcal{H}$ . We consider the operator  $T$  on  $\mathcal{F}(\mathcal{H})$  given by

$$T = X_1 + iX_2, \quad \text{with } X_1 = l(v_1) + l(v_1)^*, \quad X_2 = r(v_2) + r(v_2)^*.$$

This arises from the bi-free central limit distribution and was described in Example 3.10 of [8]. As we discussed in Section 1, we have  $[X_1, X_2] = 2i(\text{Im} \langle v_2, v_1 \rangle)P$  in the implemented  $C^*$ -probability space  $(B(\mathcal{F}(\mathcal{H})), \varphi_\Omega, P)$ , so that

$$[T, T^*] = 4(\text{Im} \langle v_2, v_1 \rangle)P. \quad (3)$$

Both the spectrum and the essential spectrum of  $X_1$  on  $\mathcal{F}(\mathcal{H})$  equal  $[-2\|v_1\|, 2\|v_1\|]$  and those of  $X_2$  equal  $[-2\|v_2\|, 2\|v_2\|]$ . By the following easy lemma, which is well known but whose proof we include for convenience, the spectrum of the operator  $T = X_1 + iX_2$  on  $\mathcal{F}(\mathcal{H})$  is contained in  $[-2\|v_1\|, 2\|v_1\|] + i[-2\|v_2\|, 2\|v_2\|]$ . Throughout this paper, we are interested in non-normal operators  $T$ ; so we assume that  $\text{Im} \langle v_2, v_1 \rangle$  is non-zero.

**Lemma 2.1.** *If  $A$  and  $B$  are self-adjoint with  $\sigma(A) \subseteq [r_1, r_2]$  and  $\sigma(B) \subseteq [t_1, t_2]$ , then  $\sigma(A + iB) \subseteq [r_1, r_2] + i[t_1, t_2]$ .*

*Proof.* If  $A_1 = A_1^*$ ,  $B_1 \geq 0$ , and  $B_1$  is invertible, then  $A_1 + iB_1 = B_1^{\frac{1}{2}} \left( B_1^{-\frac{1}{2}} A_1 B_1^{-\frac{1}{2}} + i \right) B_1^{\frac{1}{2}}$  is invertible since  $B_1^{-\frac{1}{2}} A_1 B_1^{-\frac{1}{2}}$  is self-adjoint. Suppose  $a + ib \notin [r_1, r_2] + i[t_1, t_2]$ . Then either  $a < r_1$  or  $a > r_2$  or  $b < t_1$  or  $b > t_2$ . If  $b < t_1$ , then  $A + iB - (a + ib) = (A - a) + i(B - b)$  and  $B - b \geq 0$  is invertible. So  $a + ib \notin \sigma(A + iB)$ . If  $b > t_2$ , then  $a + ib - (A + iB) = (a - A) + i(b - B)$  and  $b - B \geq 0$  is invertible, so that  $a + ib \notin \sigma(A + iB)$ . Since  $(A - a) + i(B - b) = i((B - b) - i(A - a))$ , we can easily show that  $A + iB - (a + ib)$  is invertible for each case of  $a < r_1$  and  $a > r_2$ . Therefore,  $\sigma(A + iB) \subseteq [r_1, r_2] + i[t_1, t_2]$ .  $\square$

The operator  $T \in B(\mathcal{H})$  is said to be *hyponormal*, if its self-commutator  $T^*T - TT^*$  is positive. Furthermore, if there is no reducing subspace of  $T$ , the restriction of  $T$  to which is normal, then  $T$  is said to be *pure hyponormal* or *completely non-normal hyponormal*.

**Theorem 2.2** (Theorem 2.1.3 of [6]). *Let  $T \in B(\mathcal{H})$  be a hyponormal operator with  $[T^*, T] = D$ . Then there is a unique orthogonal decomposition  $\mathcal{H} = \mathcal{H}_p(T) \oplus \mathcal{H}_n(T)$  where  $\mathcal{H}_p(T)$  and  $\mathcal{H}_n(T)$  are reducing subspaces for  $T$ , such that*

- (i)  $T_p = T|_{\mathcal{H}_p(T)}$  is pure hyponormal,
- (ii)  $T_n = T|_{\mathcal{H}_n(T)}$  is normal.

Moreover,

$$\mathcal{H}_p(T) = \bigvee \{T^{*k}T^l D(\mathcal{H}) \mid k, l \in \mathbb{N}\} \quad \text{and} \quad \mathcal{H}_n(T) = \{\zeta \in \mathcal{H} \mid DT^{*l}T^k \zeta = 0 \text{ for every } k, l \in \mathbb{N}\}.$$

As we can see in (3), if  $\text{Im} \langle v_2, v_1 \rangle \leq 0$  (or  $\geq 0$ ), then  $T = X_1 + iX_2$  is a hyponormal operator (or cohyponormal, respectively) on  $\mathcal{F}(\mathcal{H})$ . By Theorem 2.2, the pure parts  $\mathcal{H}_p(T)$  and  $\mathcal{H}_p(T^*)$  of  $T$  and  $T^*$  are equal to  $\overline{\text{alg}(T, T^*, 1)\Omega}$ .

Assuming that  $\text{Im} \langle v_2, v_1 \rangle \leq 0$ , if  $v_2$  is a scalar multiple of  $v_1$ , then  $\text{alg}(T, T^*, 1)\Omega$  is dense in  $\mathcal{F}(\mathbb{C}\langle v_1 \rangle)$  so that  $T$  is pure hyponormal on  $\mathcal{F}(\mathbb{C}\langle v_1 \rangle)$ . However, if  $v_2$  is not a scalar multiple of  $v_1$ , then  $T$  is not a pure hyponormal operator on  $\mathcal{F}(\mathbb{C}\langle v_1, v_2 \rangle)$ , that is, there exists a nontrivial reducing subspace  $\mathcal{N}$  of  $T$  in  $\mathcal{F}(\mathbb{C}\langle v_1, v_2 \rangle)$  such that  $T|_{\mathcal{N}}$  is normal. For, suppose that  $u$  is a unit vector which is orthogonal to  $v_1$  in  $\mathbb{C}\langle v_1, v_2 \rangle$  and  $v_2 = cv_1 + du$  where  $c, d \in \mathbb{C}$  are non-zero. Since  $v_2$  and  $\frac{c}{|c|^2}v_1 - \frac{d}{|d|^2}u$  are orthogonal to each other, for each  $m, n \in \mathbb{N}$ ,

$$(l(v_1) + l(v_1)^*)^m \left( u \otimes \left( \frac{c}{|c|^2}v_1 - \frac{d}{|d|^2}u \right) \right) \in \text{span} \left\{ v_1^{\otimes k} \otimes u \otimes \left( \frac{c}{|c|^2}v_1 - \frac{d}{|d|^2}u \right) \mid k \in \mathbb{N} \right\}$$

and

$$(r(v_2) + r(v_2)^*)^n \left( u \otimes \left( \frac{c}{|c|^2}v_1 - \frac{d}{|d|^2}u \right) \right) \in \text{span} \left\{ u \otimes \left( \frac{c}{|c|^2}v_1 - \frac{d}{|d|^2}u \right) \otimes v_2^{\otimes k} \mid k \in \mathbb{N} \right\}.$$

Since  $\text{alg}(T, T^*, 1) = \text{alg}(X_1, X_2, 1)$  and  $[X_1, X_2] = 2i(\text{Im} \langle v_2, v_1 \rangle)P$ ,

$$\begin{aligned} \mathcal{N} &:= \bigvee \left\{ (r(v_2) + r(v_2)^*)^n (l(v_1) + l(v_1)^*)^m \left( u \otimes \left( \frac{c}{|c|^2}v_1 - \frac{d}{|d|^2}u \right) \right) \mid m, n \in \mathbb{N} \right\} \\ &= \bigvee \left\{ v_1^{\otimes m} \otimes u \otimes \left( \frac{c}{|c|^2}v_1 - \frac{d}{|d|^2}u \right) \otimes v_2^{\otimes n} \mid m, n \in \mathbb{N} \right\}. \end{aligned}$$

is a nontrivial reducing subspace of  $T$  in  $\mathcal{F}(\mathbb{C}\langle v_1, v_2 \rangle)$  which is orthogonal to  $\mathbb{C}\Omega$ . Clearly, the restrictions of  $l(v_1) + l(v_1)^*$  and  $r(v_2) + r(v_2)^*$  to  $\mathcal{N}$  commute, so the restriction of  $T$  to  $\mathcal{N}$  is normal.

Now we will characterize the pure part  $\overline{\text{alg}(T, T^*, 1)\Omega}$  of  $T$  in  $\mathcal{F}(\mathcal{H})$  when  $v_1$  and  $v_2$  are linearly independent.

**Proposition 2.3.** *Let  $T = l(v_1) + l(v_1)^* + i(r(v_2) + r(v_2)^*)$  where  $\|v_1\| = 1$ . Suppose  $v_2 = cv_1 + du$  where  $c, d \in \mathbb{C}$  are non-zero,  $u \perp v_1$  and  $\|u\| = 1$ , and let  $w := \frac{1}{\sqrt{2}} \left( \frac{c}{|c|^2}v_1 - \frac{d}{|d|^2}u \right)$ . Let  $A_n$  be the span of length  $n$  tensor products in  $\mathcal{F}(\mathbb{C}\langle v_1, v_2 \rangle)$  for each  $n \in \mathbb{N}$  and let  $A_0 = \mathbb{C}\Omega$ . Then*

$$\overline{\text{alg}(T, T^*, 1)\Omega} = \bigoplus_{n \geq 0} (A_n \cap \text{alg}(T, T^*, 1)\Omega) \quad (4)$$

and for every  $n \in \mathbb{N}$ ,

$$B_n := \{v_1^{\otimes n}, v_1^{\otimes n-1} \otimes u, v_1^{\otimes n-2} \otimes u \otimes v_2, \dots, v_1 \otimes u \otimes v_2^{\otimes n-2}, u \otimes v_2^{\otimes n-1}\} \quad (5)$$

and

$$B'_n := \{v_2^{\otimes n}, w \otimes v_2^{\otimes n-1}, v_1 \otimes w \otimes v_2^{\otimes n-2}, \dots, v_1^{\otimes n-2} \otimes w \otimes v_2, v_1^{\otimes n-1} \otimes w\} \quad (6)$$

are orthogonal bases of  $A_n \cap \text{alg}(T, T^*, 1)\Omega$ . Furthermore, we have the obvious isomorphisms

$$\begin{aligned} \overline{\text{alg}(T, T^*, 1)\Omega} &\cong \mathcal{F}(\mathbb{C}\langle v_1 \rangle) \oplus (\mathcal{F}(\mathbb{C}\langle v_1 \rangle) \otimes u \otimes \mathcal{F}(\mathbb{C}\langle v_2 \rangle)) \\ &\cong \mathcal{F}(\mathbb{C}\langle v_2 \rangle) \oplus (\mathcal{F}(\mathbb{C}\langle v_1 \rangle) \otimes w \otimes \mathcal{F}(\mathbb{C}\langle v_2 \rangle)). \end{aligned}$$

*Proof.* We will prove by induction on  $n$  that  $B_n$  is an orthogonal basis for  $A_n \cap \text{alg}(T, T^*, 1)\Omega$ . This is clear for  $n = 1$ . For  $n = 2$ , consider the orthogonal basis of  $A_2$

$$Z_2 = \{v_1^{\otimes 2}, v_1 \otimes u, u \otimes v_2, u \otimes w\}$$

containing  $B_2$ . Here,  $B_2 = \{v_1^{\otimes 2}, v_1 \otimes u, u \otimes v_2\} \subseteq \text{alg}(T, T^*, 1)\Omega$  and  $Z_2 \setminus B_2 = \{u \otimes w\} \subseteq (\text{alg}(T, T^*, 1)\Omega)^\perp$  as we saw in the above argument describing  $\mathcal{N}$ . Now the assertion is proved for  $n = 2$ .

Consider another orthogonal basis of  $A_2$ ,  $Z'_2 = \{v_2^{\otimes 2}, v_2 \otimes w, w \otimes v_2, w^{\otimes 2}\}$ . Then  $Z_3 := \{v_1 \otimes Z_2\} \cup \{u \otimes Z'_2\}$  is an orthogonal basis of  $A_3$ . Since  $\text{alg}(T, T^*, 1)\Omega$  is a reducing subspace of  $T$ ,  $v_1 \otimes B_2 \subseteq \text{alg}(T, T^*, 1)\Omega$  and  $v_1 \otimes \{Z_2 \setminus B_2\} \subseteq (\text{alg}(T, T^*, 1)\Omega)^\perp$ . In  $u \otimes Z'_2$ , only  $u \otimes v_2^{\otimes 2}$  is contained in  $\text{alg}(T, T^*, 1)\Omega$  because every tensor product in  $\mathcal{F}(\mathbb{C}\langle v_1, v_2 \rangle)$  which starts with  $u$  and ends with  $w$  belongs to  $\mathcal{N}$  and is therefore orthogonal to  $\text{alg}(T, T^*, 1)\Omega$ ; moreover  $u \otimes w \otimes v_2 = (r(v_2) + r(v_2)^*)(u \otimes w) \in (\text{alg}(T, T^*, 1)\Omega)^\perp$ . Hence,  $B_3 = \{v_1 \otimes B_2\} \cup \{u \otimes v_2^{\otimes 2}\}$  is contained in  $\text{alg}(T, T^*, 1)\Omega$  and  $Z_3 \setminus B_3$  is contained in  $(\text{alg}(T, T^*, 1)\Omega)^\perp$ . Thus the assertion holds for  $n = 3$ .

The induction step for general  $n$  proceeds similarly. For each  $n \in \mathbb{N}$ , construct an orthogonal basis  $Z_n$  for  $A_n$  as follows.

$$Z_n = \{v_1^n\} \cup \left( \bigcup_{1 \leq j \leq n} \{v_1^{n-j} \otimes u \otimes Z'_{j-1}\} \right),$$

where  $Z'_k$  is the set of all length  $k$  tensor products in  $\mathcal{F}(\mathcal{H})$  whose components consist of  $v_2$  and  $w$ . The induction hypothesis is that  $B_j$  is an orthogonal basis of  $A_j \cap \text{alg}(T, T^*, 1)\Omega$  and  $Z_j \setminus B_j$  is orthogonal to  $\text{alg}(T, T^*, 1)\Omega$  for each  $1 \leq j \leq n$ . Then  $Z_{n+1} = \{v_1 \otimes Z_n\} \cup \{u \otimes Z'_n\}$  and it is an orthogonal basis of  $A_{n+1}$ . Since  $\text{alg}(T, T^*, 1)\Omega$  is a reducing subspace of  $T$  and is invariant under  $l(v_1) + l(v_1)^*$ , we have  $v_1 \otimes B_n = \{v_1^{\otimes n+1}, v_1^{\otimes n} \otimes u, v_1^{\otimes n-1} \otimes u \otimes v_2, \dots, v_1 \otimes u \otimes v_2^{\otimes n-1}\} \subseteq \text{alg}(T, T^*, 1)\Omega$  and  $v_1 \otimes \{Z_n \setminus B_n\} \subseteq (\text{alg}(T, T^*, 1)\Omega)^\perp$ . In  $u \otimes Z'_n$ , only  $u \otimes v_2^{\otimes n}$  is contained in  $\text{alg}(T, T^*, 1)\Omega$  and the other elements are orthogonal to  $\text{alg}(T, T^*, 1)\Omega$  by the induction hypothesis. Therefore,  $B_{n+1} = \{v_1 \otimes B_n\} \cup \{u \otimes v_2^{\otimes n}\}$  is an orthogonal basis for  $A_{n+1} \cap \text{alg}(T, T^*, 1)\Omega$  and  $Z_{n+1} \setminus B_{n+1}$  is an orthogonal basis for  $A_{n+1} \cap (\text{alg}(T, T^*, 1)\Omega)^\perp$ . Thus, for every  $n \in \mathbb{N}$ ,  $B_n$  is an orthogonal basis for the set of all length  $n$  tensor products in  $\text{alg}(T, T^*, 1)\Omega$ . This finishes the proof by induction.

The proof that for  $n \in \mathbb{N}$ ,  $B'_n$  is also an orthogonal basis for  $A_n \cap \text{alg}(T, T^*, 1)\Omega$  follows similarly by induction on  $n$ , using the invariance of  $\text{alg}(T, T^*, 1)\Omega$  under  $r(v_2) + r(v_2)^*$  rather than  $l(v_1) + l(v_1)^*$ .

The equality (4) follows by the above proofs.  $\square$

Before we further investigate the operator  $T = X_1 + iX_2$  having  $v_1$  and  $v_2$  linearly independent, we will take a look at the case when the vectors  $v_1$  and  $v_2$  are linearly dependent. We will refer to the following result.

**Theorem 2.4** ([2]). *If  $T$  is a hyponormal operator on  $\mathcal{H}$ , then  $C^*(T)$  is generated by the unilateral shift if and only if  $T$  is unitarily equivalent to  $S$ , where  $S$  satisfies conditions*

- (i)  $S$  is irreducible;
- (ii) self-commutator  $S^*S - SS^*$  is compact;
- (iii)  $\sigma_e(S)$  is a simple closed curve;
- (iv)  $\sigma(T)$  is the closure of  $V$ , where  $V$  is the bounded component of  $\mathbb{C} \setminus \sigma_e(S)$ ;
- (v) for  $\lambda \in \sigma(S) \setminus \sigma_e(S)$ ,  $\text{ind}(S - \lambda) = -1$ .

**Example 2.5.** Let  $v_1 = \alpha v_2$ ,  $\alpha \in \mathbb{C}$ ,  $\text{Im } \alpha \neq 0$ , and  $\|v_2\| = 1$ . Let  $T$  be given by

$$\begin{aligned} T &= l(v_1) + l(v_1)^* + i(r(v_2) + r(v_2)^*) \\ &= (\alpha l(v_2) + i r(v_2)) + (\bar{\alpha} l(v_2)^* + i r(v_2)^*) \end{aligned}$$

on  $\mathcal{F}(\mathbb{C})$ . Then,

$$T(\Omega) = (\alpha + i)v_2$$

and for each  $n \in \mathbb{N}$ ,

$$T(v_2^{\otimes n}) = (\alpha + i)v_2^{\otimes n+1} + (\bar{\alpha} + i)v_2^{\otimes n-1}.$$

Therefore,

$$T = (\alpha + i)U + (\bar{\alpha} + i)U^*,$$

where  $U$  is the unilateral shift on  $\mathcal{F}(\mathbb{C})$ .

If  $\alpha = i$ , then  $T = 2iU$  and  $[T^*, T] = 4P$  so that  $T$  is a hyponormal operator. If  $\alpha = -i$ , then  $T = 2iU^*$  and  $[T^*, T] = -4P$ , so  $T$  is cohyponormal.

Since the image of the unilateral shift  $U$  in the Calkin algebra is a normal operator, by the functional calculus, we have

$$\begin{aligned}\sigma_e((\alpha + i)U + (\bar{\alpha} + i)U^*) &= \{(\alpha + i)t + (\bar{\alpha} + i)\bar{t} \mid t \in \sigma_e(U)\} \\ &= \{\alpha t + \bar{\alpha}\bar{t} + i(t + \bar{t}) \mid t \in \mathbb{T}\}.\end{aligned}$$

This curve is the solution set of

$$x^2 + |\alpha|^2 y^2 - 2(\operatorname{Re} \alpha)xy = 4(\operatorname{Im} \alpha)^2 \quad (7)$$

in the  $xy$ -plane, which is an ellipse centered at the origin. So the essential spectrum of  $T$  is a simple closed curve. Let  $V_0$  be the bounded component of  $\mathbb{C} \setminus \sigma_e(T)$ . Then by Theorem 2.4, we have

$$\sigma(T) = \overline{V_0},$$

and for  $\lambda \in \sigma(T) \setminus \sigma_e(T)$ ,

$$\operatorname{ind}(T - \lambda) = \begin{cases} -1, & \operatorname{Im}(\alpha) > 0 \\ 1, & \operatorname{Im}(\alpha) < 0. \end{cases}$$

Thus, the principal function is the characteristic function of the interior of the ellipse (7) when  $\operatorname{Im} \alpha < 0$ , and is the negative of this when  $\operatorname{Im} \alpha > 0$ .

2.2. In the rest of this paper, we consider the pure part of  $T = X_1 + iX_2$  acting on  $\overline{\operatorname{alg}(T, T^*, 1)\Omega}$  where  $X_1 = l(v_1) + l(v_1)^*$  and  $X_2 = r(v_2) + r(v_2)^*$ . So  $T$  is a completely non-normal operator.

Now we will find a formula for the principal function of  $T$  when  $v_1$  and  $v_2$  are linearly independent. For this, we will use equation (2); so we will first establish a formula for  $\det E(l, s)$  of  $T$ . Suppose  $l \in \mathbb{C} \setminus \sigma(X_2)$  and  $s \in \mathbb{C} \setminus \sigma(X_1)$ . From (1), we have

$$\begin{aligned}\det E(l, s) &= \det((X_2 - l)(X_1 - s)(X_2 - l)^{-1}(X_1 - s)^{-1}) \\ &= \det(((X_1 - s)(X_2 - l) - 2\operatorname{Im} \langle v_2, v_1 \rangle iP)(X_2 - l)^{-1}(X_1 - s)^{-1}) \\ &= \det(1 - 2\operatorname{Im} \langle v_2, v_1 \rangle iP(X_2 - l)^{-1}(X_1 - s)^{-1}) \\ &= \det(1 - 2\operatorname{Im} \langle v_2, v_1 \rangle iP^2(X_2 - l)^{-1}(X_1 - s)^{-1}) \\ &= \det(1 - 2\operatorname{Im} \langle v_2, v_1 \rangle iP(X_2 - l)^{-1}(X_1 - s)^{-1}P) \\ &= \det(1 - 2\operatorname{Im} \langle v_2, v_1 \rangle i\varphi_\Omega((X_2 - l)^{-1}(X_1 - s)^{-1})P) \\ &= 1 - 2\operatorname{Im} \langle v_2, v_1 \rangle i\varphi_\Omega((X_2 - l)^{-1}(X_1 - s)^{-1}) \\ &= 1 - 2\operatorname{Im} \langle v_2, v_1 \rangle i\overline{\varphi_\Omega((\bar{s} - X_1)^{-1}(\bar{l} - X_2)^{-1})} \\ &= 1 - 2\operatorname{Im} \langle v_2, v_1 \rangle i\overline{G_{(X_1, X_2)}(\bar{s}, \bar{l})} \quad (8)\end{aligned}$$

where  $G_{(X_1, X_2)}(z, w) = \varphi((z - X_1)^{-1}(w - X_2)^{-1})$ . Note that  $G_{(X_1, X_2)}(z, w)$  is the germ of a holomorphic function near  $(\infty, \infty)$  in  $\mathbb{C}_\infty \times \mathbb{C}_\infty$  (see [8]).

2.3. We review the definition and formula of the partial bi-free R-transform,  $R_{(a,b)}(z, w)$ , defined in [8] and find  $\det E(l, s)$  in terms of  $l$  and  $s$ .

**Definition 2.6** ([8]). Let  $(a, b)$  be a two-faced pair of non-commutative random variables in  $(\mathcal{A}, \varphi)$ . Set  $I = \{i\}$  and  $J = \{j\}$  and suppose  $\alpha : \{1, \dots, m+n\} \rightarrow I \amalg J$  is given by  $\alpha(k) = i$  if  $1 \leq k \leq m$  and  $\alpha(k) = j$  if  $m+1 \leq k \leq m+n$ . We shall denote the bi-free cummulant  $R_\alpha$  as  $R_{m,n}$ . The *partial bi-free R-transform* is the generating series

$$R_{(a,b)}(z, w) = \sum_{\substack{m \geq 0, n \geq 0 \\ m+n \geq 1}} R_{m,n}(a, b) z^m w^n.$$

**Theorem 2.7** (Theorem 2.4 of [8]). *We have the equality of germs of holomorphic functions near  $(0, 0) \in \mathbb{C}^2$ ,*

$$R_{(a,b)}(z, w) = 1 + zR_a(z) + wR_b(w) - \frac{zw}{G_{(a,b)}(K_a(z), K_b(w))},$$

where  $R_a(z)$  and  $R_b(w)$  are one variable  $R$ -transforms, and  $K_a(z) = z^{-1} + R_a(z)$  and  $K_b(w) = w^{-1} + R_b(w)$ .

For the given two-faced pair  $(X_1, X_2)$ , the definition of the partial bi-free  $R$ -transform and Lemma 7.2 of [7] give

$$\begin{aligned} R_{(X_1, X_2)}(z, w) &= R_{2,0}(X_1, X_2) + R_{0,2}(X_1, X_2) + R_{1,1}(X_1, X_2) \\ &= \varphi(X_1^2)z^2 + \varphi(X_2^2)w^2 + \varphi(X_1 X_2)zw \\ &= \|v_1\|^2 z^2 + \|v_2\|^2 w^2 + \langle v_2, v_1 \rangle zw. \end{aligned} \quad (9)$$

From the formula for the partial bi-free  $R$ -transform in Theorem 2.7, we also have

$$R_{(X_1, X_2)}(z, w) = 1 + \|v_1\|^2 z^2 + \|v_2\|^2 w^2 - \frac{zw}{G_{(X_1, X_2)}(\frac{1}{z} + \|v_1\|^2 z, \frac{1}{w} + \|v_2\|^2 w)}. \quad (10)$$

Denoting

$$t_1 = \frac{1}{z} + \|v_1\|^2 z \quad \text{and} \quad t_2 = \frac{1}{w} + \|v_2\|^2 w,$$

for  $z, w \in \mathbb{C} \setminus \{0\}$  close to 0, we have

$$z = \frac{t_1 - \sqrt{t_1^2 - 4\|v_1\|^2}}{2\|v_1\|^2} \quad \text{and} \quad w = \frac{t_2 - \sqrt{t_2^2 - 4\|v_2\|^2}}{2\|v_2\|^2},$$

where the branches of the square roots are  $\sqrt{t_1^2 - 4\|v_1\|^2} \approx t_1$  and  $\sqrt{t_2^2 - 4\|v_2\|^2} \approx t_2$  for  $|t_1|$  and  $|t_2|$  large. From the formulas (9) and (10), we get

$$\begin{aligned} G_{(X_1, X_2)}(t_1, t_2) &= \frac{zw}{1 - \langle v_2, v_1 \rangle zw} \\ &= \frac{(t_1 - \sqrt{t_1^2 - 4\|v_1\|^2})(t_2 - \sqrt{t_2^2 - 4\|v_2\|^2})}{4\|v_1\|^2\|v_2\|^2 - \langle v_2, v_1 \rangle (t_1 - \sqrt{t_1^2 - 4\|v_1\|^2})(t_2 - \sqrt{t_2^2 - 4\|v_2\|^2})}, \end{aligned} \quad (11)$$

when  $|t_1|$  and  $|t_2|$  are large.

Let

$$q(t) = \frac{t - \sqrt{t^2 - 4}}{2} \quad (t \in \mathbb{C} \setminus [-2, 2]). \quad (12)$$

The function  $z \mapsto z + \frac{1}{z}$  sends the punctured unit disk  $\{z \mid 0 < |z| < 1\}$  biholomorphically onto  $\mathbb{C} \setminus [-2, 2]$ . The function  $q$  is its inverse with respect to composition. We deduce that the identity  $q(t) = \overline{q(\bar{t})}$  holds for all  $t \in \mathbb{C} \setminus [-2, 2]$ .

By (8) and (11), for  $|l|$  and  $|s|$  large, we have

$$\begin{aligned} \det E(l, s) &= 1 - 2(\operatorname{Im} \langle v_2, v_1 \rangle) i \left( \frac{(\bar{s} - \sqrt{\bar{s}^2 - 4\|v_1\|^2})(\bar{l} - \sqrt{\bar{l}^2 - 4\|v_2\|^2})}{4\|v_1\|^2\|v_2\|^2 - \langle v_2, v_1 \rangle (\bar{s} - \sqrt{\bar{s}^2 - 4\|v_1\|^2})(\bar{l} - \sqrt{\bar{l}^2 - 4\|v_2\|^2})} \right) \\ &= 1 + \frac{2(\operatorname{Im} \langle v_1, v_2 \rangle) i (s - \sqrt{s^2 - 4\|v_1\|^2})(l - \sqrt{l^2 - 4\|v_2\|^2})}{4\|v_1\|^2\|v_2\|^2 - \langle v_1, v_2 \rangle (s - \sqrt{s^2 - 4\|v_1\|^2})(l - \sqrt{l^2 - 4\|v_2\|^2})}, \end{aligned}$$

$$\begin{aligned}
&= \frac{4\|v_1\|^2\|v_2\|^2 - \overline{\langle v_1, v_2 \rangle}(s - \sqrt{s^2 - 4\|v_1\|^2})(l - \sqrt{l^2 - 4\|v_2\|^2})}{4\|v_1\|^2\|v_2\|^2 - \langle v_1, v_2 \rangle(s - \sqrt{s^2 - 4\|v_1\|^2})(l - \sqrt{l^2 - 4\|v_2\|^2})} \\
&= \frac{1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|}q\left(\frac{s}{\|v_1\|}\right)q\left(\frac{l}{\|v_2\|}\right)}{1 - \frac{\alpha}{\|v_1\|\|v_2\|}q\left(\frac{s}{\|v_1\|}\right)q\left(\frac{l}{\|v_2\|}\right)}, \tag{13}
\end{aligned}$$

where  $\alpha = \langle v_1, v_2 \rangle$ , and for the second equality, we have used

$$\overline{(\bar{s} - \sqrt{\bar{s}^2 - 4\|v_1\|^2})} = 2\|v_1\|q\left(\frac{\bar{s}}{\|v_1\|}\right) = 2\|v_1\|q\left(\frac{s}{\|v_1\|}\right) = s - \sqrt{s^2 - 4\|v_1\|^2}$$

and

$$\overline{(\bar{l} - \sqrt{\bar{l}^2 - 4\|v_2\|^2})} = 2\|v_2\|q\left(\frac{\bar{l}}{\|v_2\|}\right) = 2\|v_2\|q\left(\frac{l}{\|v_2\|}\right) = l - \sqrt{l^2 - 4\|v_2\|^2}.$$

Since  $v_1$  and  $v_2$  are linearly independent,  $|\alpha| < \|v_1\|\|v_2\|$ . Since  $\left|q\left(\frac{s}{\|v_1\|}\right)\right| < 1$  and  $\left|q\left(\frac{l}{\|v_2\|}\right)\right| < 1$  for  $s \in \mathbb{C} \setminus [-2\|v_1\|, 2\|v_1\|]$  and  $l \in \mathbb{C} \setminus [-2\|v_2\|, 2\|v_2\|]$ , the numerator and denominator in (13) do not vanish for such  $s$  and  $l$ . So the right-hand side of (13) is a holomorphic function there. Since by definition in (8),  $\det E(l, s)$  is holomorphic on  $(\mathbb{C}_\infty \setminus \sigma(X_2)) \times (\mathbb{C}_\infty \setminus \sigma(X_1))$ , it follows from the analytic continuation that the formula of  $\det E(l, s)$  in (13) holds for all  $s \in \mathbb{C} \setminus [-2\|v_1\|, 2\|v_1\|]$  and  $l \in \mathbb{C} \setminus [-2\|v_2\|, 2\|v_2\|]$ .

2.4. In this subsection, we find the formula of the principal function  $g(\delta, \gamma)$  of  $T$  by using the formula (13). The principal function  $g$  was defined on  $\mathbb{R}^2$  by

$$\det E(l, s) = \exp\left(\frac{1}{2\pi i} \int_{\mathbb{R}} \int_{\mathbb{R}} g(\delta, \gamma) \frac{d\delta}{\delta - l} \frac{d\gamma}{\gamma - s}\right)$$

and  $\text{supp}(g) \subseteq \{(\delta, \gamma) \in \mathbb{R}^2 \mid \gamma + i\delta \in \sigma(T)\}$ . To find the principal function of  $T$ , consider the function  $f$  defined by

$$f(l, \gamma) = \int_{\mathbb{R}} g(\delta, \gamma) \frac{d\delta}{\delta - l}.$$

for  $l \in \mathbb{C} \setminus \sigma(X_2)$  and  $\gamma \in \mathbb{R}$ . Fixing  $\gamma \in \mathbb{R}$ ,  $f(l, \gamma)$  is a holomorphic function for  $l \in \mathbb{C} \setminus \sigma(X_2)$ . From (13) and the definition of  $g(\delta, \gamma)$ , we have

$$\int_{\mathbb{R}} f(l, \gamma) \frac{d\gamma}{\gamma - s} = (2\pi i) \text{Log} \left( \frac{1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|}q\left(\frac{s}{\|v_1\|}\right)q\left(\frac{l}{\|v_2\|}\right)}{1 - \frac{\alpha}{\|v_1\|\|v_2\|}q\left(\frac{s}{\|v_1\|}\right)q\left(\frac{l}{\|v_2\|}\right)} \right), \tag{14}$$

where  $s \in \mathbb{C}_\infty \setminus \sigma(X_1)$  and  $l \in \mathbb{C}_\infty \setminus \sigma(X_2)$ . Now we will find the function  $f(l, \gamma)$  by using the *Stieltjes inversion formula*.

We defined the function  $q(t)$  for  $t \in \mathbb{C} \setminus [-2, 2]$  in (12).

**Lemma 2.8.** *If  $t_0 \in [-2, 2]$ , then*

$$\lim_{\epsilon \searrow 0} q(t_0 + i\epsilon) = \frac{t_0 - i\sqrt{4 - t_0^2}}{2}.$$

*Proof.* For  $t_0 \in (-2, 2)$ ,

$$\begin{aligned}
\lim_{\epsilon \searrow 0} q(t_0 + i\epsilon) &= \lim_{\epsilon \searrow 0} \frac{t_0 + i\epsilon - \sqrt{(t_0 + i\epsilon)^2 - 4}}{2} \\
&= \lim_{\epsilon \searrow 0} \frac{t_0 + i\epsilon - \sqrt{-(4 + \epsilon^2 - t_0^2) + 2i\epsilon t_0}}{2} \\
&= \frac{t_0 - i\sqrt{4 - t_0^2}}{2}.
\end{aligned}$$



For, when  $\epsilon$  is large and positive, the branch of a square root is such that  $\sqrt{-(4 + \epsilon^2 - t_0^2) + 2i\epsilon t_0} \approx t_0 + i\epsilon$ . So  $\lim_{\epsilon \searrow 0} \sqrt{-(4 + \epsilon^2 - t_0^2) + 2i\epsilon t_0} = i\sqrt{4 - t_0^2}$ .  $\square$

Define a function  $\zeta(t)$  for  $t \in [-2, 2]$  by

$$\zeta(t) = \frac{t - i\sqrt{4 - t^2}}{2}.$$

Then  $\zeta(t) \in \mathbb{T}$  for  $t \in [-2, 2]$ , where  $\mathbb{T}$  is a unit circle in  $\mathbb{C}$ . By Lemma 2.8, the limit of  $q(t + i\epsilon)$  goes to  $\zeta(t)$  as  $\epsilon \searrow 0$ , where  $t \in [-2, 2]$ . Then we have for  $\gamma \in [-2\|v_1\|, 2\|v_1\|]$ ,

$$\lim_{\epsilon \searrow 0} q\left(\frac{\gamma}{\|v_1\|} + i\frac{\epsilon}{\|v_1\|}\right) = \frac{\frac{\gamma}{\|v_1\|} - i\sqrt{4 - \left(\frac{\gamma}{\|v_1\|}\right)^2}}{2} = \zeta\left(\frac{\gamma}{\|v_1\|}\right) \in \mathbb{T}. \quad (15)$$

Fix  $l \in \mathbb{R} \setminus [-2\|v_2\|, 2\|v_2\|]$ . Since clearly  $f(l, \gamma) = 0$  for  $\gamma \in \mathbb{R} \setminus \sigma(X_1)$ , we suppose  $\gamma \in \sigma(X_1)$ . Using (14), the Stieltjes inversion formula, and (15), we have

$$\begin{aligned} f(l, \gamma) &= \frac{1}{\pi} \lim_{\epsilon \searrow 0} \operatorname{Im} \left( (2\pi i) \operatorname{Log} \left( \frac{1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|} q\left(\frac{\gamma}{\|v_1\|} + i\frac{\epsilon}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)}{1 - \frac{\alpha}{\|v_1\|\|v_2\|} q\left(\frac{\gamma}{\|v_1\|} + i\frac{\epsilon}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)} \right) \right) \\ &= 2 \log \left| \frac{1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|} \zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)}{1 - \frac{\alpha}{\|v_1\|\|v_2\|} \zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)} \right| \\ &= \log \frac{\left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|} \zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)\right) \left(1 - \frac{\alpha}{\|v_1\|\|v_2\|} \overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} q\left(\frac{l}{\|v_2\|}\right)\right)}{\left(1 - \frac{\alpha}{\|v_1\|\|v_2\|} \zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)\right) \left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|} \overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} q\left(\frac{l}{\|v_2\|}\right)\right)} \\ &= \operatorname{Log} \left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|} \zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)\right) + \operatorname{Log} \left(1 - \frac{\alpha}{\|v_1\|\|v_2\|} \overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} q\left(\frac{l}{\|v_2\|}\right)\right) \\ &\quad - \operatorname{Log} \left(1 - \frac{\alpha}{\|v_1\|\|v_2\|} \zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)\right) - \operatorname{Log} \left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|} \overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} q\left(\frac{l}{\|v_2\|}\right)\right), \quad (17) \end{aligned}$$

where  $\operatorname{Log}$  is the principal branch of the logarithm. This equality holds where  $\gamma \in \sigma(X_1)$  and  $l \in \mathbb{R} \setminus [-2\|v_2\|, 2\|v_2\|]$ .

Fix  $\gamma \in \sigma(X_1)$ . Since each expression appearing as an argument of  $\operatorname{Log}$ , above, remains in the disk of radius 1 centered at 1 for  $l \in \mathbb{C} \setminus [-2\|v_2\|, 2\|v_2\|]$ . So the expression (17) is holomorphic on  $\mathbb{C} \setminus [-2\|v_2\|, 2\|v_2\|]$ . The equality (16) was derived for  $l \in \mathbb{R} \setminus [-2\|v_2\|, 2\|v_2\|]$ , but as defined,  $f(l, \gamma)$  is holomorphic in  $\mathbb{C} \setminus [-2\|v_2\|, 2\|v_2\|]$ . By analytic continuation, we have

$$\begin{aligned} &\int_{\mathbb{R}} g(\delta, \gamma) \frac{d\delta}{\delta - l} \\ &= \operatorname{Log} \left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|} \zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)\right) + \operatorname{Log} \left(1 - \frac{\alpha}{\|v_1\|\|v_2\|} \overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} q\left(\frac{l}{\|v_2\|}\right)\right) \\ &\quad - \operatorname{Log} \left(1 - \frac{\alpha}{\|v_1\|\|v_2\|} \zeta\left(\frac{\gamma}{\|v_1\|}\right) q\left(\frac{l}{\|v_2\|}\right)\right) - \operatorname{Log} \left(1 - \frac{\bar{\alpha}}{\|v_1\|\|v_2\|} \overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} q\left(\frac{l}{\|v_2\|}\right)\right) \end{aligned}$$

for  $\gamma \in \sigma(X_1)$  and  $l \in \mathbb{C} \setminus \sigma(X_2)$ .

Now we will apply the Stieltjes inversion formula to  $f(l, \gamma)$  in order to recover the principal function  $g(\delta, \gamma)$  of  $T$ . Since we have  $\lim_{\epsilon \searrow 0} q\left(\frac{\delta}{\|v_2\|} + i\frac{\epsilon}{\|v_2\|}\right) = \zeta\left(\frac{\delta}{\|v_2\|}\right)$  for  $\delta \in \sigma(X_2)$  as in (15), we get

$$\begin{aligned} g(\delta, \gamma) &= \frac{1}{\pi} \lim_{\epsilon \searrow 0} \operatorname{Im} f(\delta + i\epsilon, \gamma) \\ &= \frac{1}{\pi} \operatorname{Arg} \left( 1 - \frac{\bar{\alpha}}{\|v_1\| \|v_2\|} \zeta\left(\frac{\gamma}{\|v_1\|}\right) \zeta\left(\frac{\delta}{\|v_2\|}\right) \right) + \frac{1}{\pi} \operatorname{Arg} \left( 1 - \frac{\alpha}{\|v_1\| \|v_2\|} \overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} \zeta\left(\frac{\delta}{\|v_2\|}\right) \right) \\ &\quad - \frac{1}{\pi} \operatorname{Arg} \left( 1 - \frac{\alpha}{\|v_1\| \|v_2\|} \zeta\left(\frac{\gamma}{\|v_1\|}\right) \zeta\left(\frac{\delta}{\|v_2\|}\right) \right) - \frac{1}{\pi} \operatorname{Arg} \left( 1 - \frac{\bar{\alpha}}{\|v_1\| \|v_2\|} \overline{\zeta\left(\frac{\gamma}{\|v_1\|}\right)} \zeta\left(\frac{\delta}{\|v_2\|}\right) \right). \end{aligned}$$

Setting  $\frac{\alpha}{\|v_1\| \|v_2\|} = re^{i\phi}$  ( $0 < r < 1$ ),  $\zeta\left(\frac{\gamma}{\|v_1\|}\right) = e^{i\theta_1}$ , and  $\zeta\left(\frac{\delta}{\|v_2\|}\right) = e^{i\theta_2}$ , we get

$$\begin{aligned} g(\delta, \gamma) &= \frac{1}{\pi} \left( \operatorname{Arg}(1 - re^{i(-\phi + \theta_1 + \theta_2)}) + \operatorname{Arg}(1 - re^{i(\phi - \theta_1 + \theta_2)}) \right. \\ &\quad \left. - \operatorname{Arg}(1 - re^{i(\phi + \theta_1 + \theta_2)}) - \operatorname{Arg}(1 - re^{i(-\phi - \theta_1 + \theta_2)}) \right) \\ &= \frac{1}{\pi} \left( \arctan \left( \frac{-r \sin(-\phi + (\theta_1 + \theta_2))}{1 - r \cos(-\phi + (\theta_1 + \theta_2))} \right) + \arctan \left( \frac{-r \sin(\phi - (\theta_1 - \theta_2))}{1 - r \cos(\phi - (\theta_1 - \theta_2))} \right) \right. \\ &\quad \left. - \arctan \left( \frac{-r \sin(\phi + (\theta_1 + \theta_2))}{1 - r \cos(\phi + (\theta_1 + \theta_2))} \right) - \arctan \left( \frac{-r \sin(-\phi - (\theta_1 - \theta_2))}{1 - r \cos(-\phi - (\theta_1 - \theta_2))} \right) \right) \\ &= \frac{1}{\pi} \left( \arctan \left( \frac{r \sin(\phi - (\theta_1 + \theta_2))}{1 - r \cos(\phi - (\theta_1 + \theta_2))} \right) + \arctan \left( \frac{r \sin(\phi + (\theta_1 + \theta_2))}{1 - r \cos(\phi + (\theta_1 + \theta_2))} \right) \right. \\ &\quad \left. - \arctan \left( \frac{r \sin(\phi - (\theta_1 - \theta_2))}{1 - r \cos(\phi - (\theta_1 - \theta_2))} \right) - \arctan \left( \frac{r \sin(\phi + (\theta_1 - \theta_2))}{1 - r \cos(\phi + (\theta_1 - \theta_2))} \right) \right). \end{aligned} \tag{18}$$

### 3. ON THE ESSENTIAL SPECTRUM

As an application, we determine the essential spectrum of the operator  $T$  whose principal function we found in Section 2. We will use the following, which follows from Theorem 8.1 of [4]:

**Theorem 3.1** ([4]). *Suppose  $T$  is an operator on a Hilbert space  $\mathcal{H}$  with self-commutator  $T^*T - TT^*$  in trace class. For  $\gamma + i\delta$  not in the essential spectrum of  $T$ ,*

$$g(\delta, \gamma) = \operatorname{ind}(T - (\gamma + i\delta)),$$

where  $g(\delta, \gamma)$  is the principal function for  $T$ .

**Lemma 3.2.** *Let  $0 < r < 1$  and let*

$$\begin{aligned} h(r, \phi, \theta_1, \theta_2) &= \frac{1}{\pi} \left( \arctan \left( \frac{r \sin(\phi - (\theta_1 + \theta_2))}{1 - r \cos(\phi - (\theta_1 + \theta_2))} \right) + \arctan \left( \frac{r \sin(\phi + (\theta_1 + \theta_2))}{1 - r \cos(\phi + (\theta_1 + \theta_2))} \right) \right. \\ &\quad \left. - \arctan \left( \frac{r \sin(\phi - (\theta_1 - \theta_2))}{1 - r \cos(\phi - (\theta_1 - \theta_2))} \right) - \arctan \left( \frac{r \sin(\phi + (\theta_1 - \theta_2))}{1 - r \cos(\phi + (\theta_1 - \theta_2))} \right) \right). \end{aligned}$$

(a) *If  $\phi = 0$  or  $\phi = \pi$ , then  $h(r, \phi, \theta_1, \theta_2) = 0$ .*

(b) *If  $0 < \phi < \pi$ , then for all  $\theta_1, \theta_2 \in [-\pi, 0]$ , we have*

$$-\frac{2}{\pi} \arctan \left( \frac{2r \sin \phi}{1 - r^2} \right) \leq h(r, \phi, \theta_1, \theta_2) \leq 0,$$

with equality holding on the left when  $\theta_1 = \theta_2 = -\frac{\pi}{2}$  and equality holding on the right only when  $\theta_1 \in \{-\pi, 0\}$  or  $\theta_2 \in \{-\pi, 0\}$ .

(c) If  $\pi < \phi < 2\pi$ , then for all  $\theta_1, \theta_2 \in [-\pi, 0]$ , we have

$$0 \leq h(r, \phi, \theta_1, \theta_2) \leq -\frac{2}{\pi} \arctan\left(\frac{2r \sin \phi}{1 - r^2}\right),$$

with equality holding on the right when  $\theta_1 = \theta_2 = -\frac{\pi}{2}$  and equality holding on the left only when  $\theta_1 \in \{-\pi, 0\}$  or  $\theta_2 \in \{-\pi, 0\}$ .

*Proof.* Part (a) is clear and we may assume  $\phi \in (0, \pi) \cup (\pi, 2\pi)$ .

Let  $\nu = \theta_1 + \theta_2$  and  $\mu = \theta_1 - \theta_2$ . Then we are interested in the function

$$\begin{aligned} \tilde{h}(r, \phi, \nu, \mu) = & \frac{1}{\pi} \left( \arctan\left(\frac{r \sin(\phi - \nu)}{1 - r \cos(\phi - \nu)}\right) + \arctan\left(\frac{r \sin(\phi + \nu)}{1 - r \cos(\phi + \nu)}\right) \right. \\ & \left. - \arctan\left(\frac{r \sin(\phi - \mu)}{1 - r \cos(\phi - \mu)}\right) - \arctan\left(\frac{r \sin(\phi + \mu)}{1 - r \cos(\phi + \mu)}\right) \right), \end{aligned}$$

where

$$-2\pi \leq \nu \leq 0 \tag{19}$$

$$-\min(-\nu, 2\pi + \nu) \leq \mu \leq \min(-\nu, 2\pi + \nu). \tag{20}$$

In particular, we always have  $|\mu| \leq \pi$ . Note that the boundaries of the region described by (19)–(20) correspond to  $\theta_1 \in \{-\pi, 0\}$  or  $\theta_2 \in \{-\pi, 0\}$ , where the function  $h$  vanishes.

An extreme point of  $\tilde{h}$  not on the boundary can occur only where

$$\frac{\partial \tilde{h}}{\partial \nu} = \frac{\partial \tilde{h}}{\partial \mu} = 0.$$

We compute

$$\frac{d}{dx} \arctan\left(\frac{r \sin(x)}{1 - r \cos(x)}\right) = \frac{r(\cos(x) - r)}{1 - 2r \cos(x) + r^2}.$$

We also compute

$$\frac{d}{dc} \left( \frac{c - r}{1 - 2rc + r^2} \right) = \frac{1 - r^2}{(1 - 2rc + r^2)^2} > 0,$$

so the function

$$c \mapsto \frac{r(c - r)}{1 - 2rc + r^2}$$

is strictly increasing on  $[-1, 1]$ . Therefore,

$$\begin{aligned} \frac{\partial \tilde{h}}{\partial \nu} = & \frac{d}{d\nu} \left( \arctan\left(\frac{r \sin(\phi - \nu)}{1 - r \cos(\phi - \nu)}\right) + \arctan\left(\frac{r \sin(\phi + \nu)}{1 - r \cos(\phi + \nu)}\right) \right) \\ = & \frac{-r(\cos(\phi - \nu) - r)}{1 - 2r \cos(\phi - \nu) + r^2} + \frac{r(\cos(\phi + \nu) - r)}{1 - 2r \cos(\phi + \nu) + r^2} \end{aligned}$$

vanishes if and only if  $\cos(\phi - \nu) = \cos(\phi + \nu)$ , which in turn occurs if and only if either  $\nu \in \pi\mathbf{Z}$  or  $\phi \in \pi\mathbf{Z}$ . We assumed  $\phi \notin \pi\mathbf{Z}$ . If  $\nu \in \{-2\pi, 0\}$ , then  $\nu$  is on the boundary of the interval (19), so the only possibility that is not on the boundary of the region is  $\nu = -\pi$ .

Arguing as above,  $\frac{\partial \tilde{h}}{\partial \mu} = 0$  if and only if  $\cos(\phi - \mu) = \cos(\phi + \mu)$ . Avoiding the boundary, this leaves only  $\mu = 0$ . We conclude that the only extreme point of  $\tilde{h}$  not on the boundary occurs at  $(\nu, \mu) = (-\pi, 0)$ , i.e., at  $(\theta_1, \theta_2) = (-\frac{\pi}{2}, -\frac{\pi}{2})$ , and the value of  $\tilde{h}$  there is

$$-\frac{2}{\pi} \left( \arctan\left(\frac{r \sin \phi}{1 - r \cos \phi}\right) + \arctan\left(\frac{r \sin \phi}{1 + r \cos \phi}\right) \right). \tag{21}$$

We have the identity, for  $\alpha, \beta \in \mathbb{R}$ ,

$$\arctan(\alpha) + \arctan(\beta) \in \arctan\left(\frac{\alpha + \beta}{1 - \alpha\beta}\right) + \pi\mathbf{Z}.$$

Letting

$$\alpha = \frac{r \sin \phi}{1 - r \cos \phi} \quad \text{and} \quad \beta = \frac{r \sin \phi}{1 + r \cos \phi},$$

since

$$0 < \alpha\beta = \frac{r^2 \sin^2 \phi}{1 - r^2 + r^2 \sin^2 \phi} < 1,$$

we find that the quantity (21) equals

$$-\frac{2}{\pi} \arctan\left(\frac{\alpha + \beta}{1 - \alpha\beta}\right) = -\frac{2}{\pi} \arctan\left(\frac{2r \sin \phi}{1 - r^2}\right). \quad (22)$$

We already observed that on the boundaries of the region described by (19)–(20), the function  $\tilde{h}$  vanishes and we just showed that the only extreme value not on the boundary is (22), which is attained when  $\theta_1 = \theta_2 = -\frac{\pi}{2}$ . In particular,  $\tilde{h}$  is never vanishing on the interior of the region. This completes the proof of (b) and (c).  $\square$

**Theorem 3.3.** *Let  $T = l(v_1) + l(v_1)^* + i(r(v_2) + r(v_2)^*)$  with  $v_1$  and  $v_2$  linearly independent and  $\text{Im} \langle v_1, v_2 \rangle \neq 0$ . Then the essential spectrum  $\sigma_e(T)$  of  $T$  is the closed rectangle*

$$\{\gamma + i\delta \in \mathbb{C} \mid |\gamma| \leq 2\|v_1\| \text{ and } |\delta| \leq 2\|v_2\|\}, \quad (23)$$

which equals the spectrum  $\sigma(T)$  of  $T$ .

*Proof.* By Lemma 2.1, we have that  $\sigma(T)$  is contained in the rectangle (23). For  $\gamma \in \sigma(X_1)$  and  $\delta \in \sigma(X_2)$ , we have the formula of the principal function  $g(\delta, \gamma)$  in (18). By Lemma 3.2,  $-1 < g(\delta, \gamma) \leq 0$  if  $\text{Im} \langle v_1, v_2 \rangle > 0$ , and  $0 \leq g(\delta, \gamma) < 1$  if  $\text{Im} \langle v_1, v_2 \rangle < 0$ . The equality  $g(\delta, \gamma) = 0$  holds only when  $\gamma \in \{2\|v_1\|, -2\|v_1\|\}$  or  $\delta \in \{2\|v_2\|, -2\|v_2\|\}$ , i.e., when  $\gamma$  and  $\delta$  are on the boundary of the rectangle (23). So the function  $g(\delta, \gamma)$  does not assume any integer value on the interior of the rectangle. But, by Theorem 3.1, if  $\gamma + i\delta \notin \sigma_e(T)$ , then  $g(\delta, \gamma) = \text{ind}(T - (\gamma + i\delta))$ . So the whole interior of the rectangle is included in the essential spectrum of  $T$ . Since  $\sigma_e(T)$  is closed in  $\mathbb{C}$  and is contained in  $\sigma(T)$ , we have  $\sigma_e(T)$  equals the rectangle (23).  $\square$

**Proposition 3.4** ([1]). *Suppose that  $T$  has compact self-commutator  $T^*T - TT^*$  on a Hilbert space  $\mathcal{H}$  and  $\text{ind}(T - \lambda) = 0$  for all  $\lambda \in \mathbb{C} \setminus \sigma_e(T)$ . Then  $T$  is of the form  $N + K$  where  $N$  is normal and  $K$  is compact.*

**Corollary 3.5.** *The operator  $T = l(v_1) + l(v_1)^* + i(r(v_2) + r(v_2)^*)$  with linearly independent  $v_1$  and  $v_2$  and  $\text{Im} \langle v_1, v_2 \rangle \neq 0$  is normal plus compact.*

**Example 3.6.** Let  $v_2$  and  $u$  be orthogonal vectors in a Hilbert space  $\mathcal{H}$  with  $\|v_2\| = \|u\| = 1$ , and let  $\alpha = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \in \mathbb{C}$ . Set  $v_1$  in  $\mathcal{H}$  by  $v_1 = \alpha v_2 + u$ . Suppose that  $T$  is a bounded operator on the full Fock space  $\mathcal{F}(\mathcal{H})$  defined by  $T = l(v_1) + l(v_1)^* + i(r(v_2) + r(v_2)^*)$ . Then,  $[T, T^*] = 2\sqrt{2}P$  and it is an one-dimensional projection on  $\mathcal{F}(\mathcal{H})$ . So, by restricting  $T^*$  to its pure part  $\text{alg}(T, T^*, 1)\Omega$ ,  $T^*$  is a completely non-normal hyponormal operator.

We can find the principal function  $g(\delta, \gamma)$  of  $T$  by the formula (18). For each pair  $(\delta, \gamma)$  such that  $|\delta| \leq 2$  and  $|\gamma| \leq 2\sqrt{2}$ , we have

$$\begin{aligned} g(\delta, \gamma) = & \frac{1}{\pi} \text{Arg} \left( 1 - \left( \frac{1}{2} + \frac{1}{2}i \right) \zeta \left( \frac{\gamma}{\sqrt{2}} \right) \zeta(\delta) \right) + \frac{1}{\pi} \text{Arg} \left( 1 - \left( \frac{1}{2} - \frac{1}{2}i \right) \zeta \left( \frac{\gamma}{\sqrt{2}} \right) \zeta(\delta) \right) \\ & - \frac{1}{\pi} \text{Arg} \left( 1 - \left( \frac{1}{2} - \frac{1}{2}i \right) \zeta \left( \frac{\gamma}{\sqrt{2}} \right) \zeta(\delta) \right) - \frac{1}{\pi} \text{Arg} \left( 1 - \left( \frac{1}{2} + \frac{1}{2}i \right) \zeta \left( \frac{\gamma}{\sqrt{2}} \right) \zeta(\delta) \right), \end{aligned}$$

where  $\zeta(t) = \frac{t-i\sqrt{4-t^2}}{2}$  for  $t \in [-2, 2]$ . Since  $\text{Im} \langle v_1, v_2 \rangle < 0$ , we have  $0 \leq g(\delta, \gamma) < 1$  for all  $(\delta, \gamma) \in \mathbb{R}^2$ . By Lemma 3.2,  $g(\delta, \gamma)$  is vanishing only when  $(\delta, \gamma)$  is on the boundary of the rectangle  $\{(\delta, \gamma) \in \mathbb{R}^2 \mid |\gamma| \leq 2\sqrt{2} \text{ and } |\delta| \leq 2\}$ . Therefore,  $\sigma(T) = \sigma_e(T) = \{\gamma + i\delta \in \mathbb{C} \mid |\gamma| \leq 2\sqrt{2} \text{ and } |\delta| \leq 2\}$ . See Figure 1.

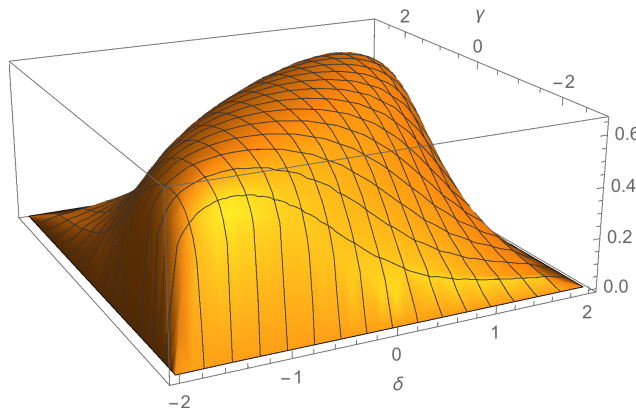


FIGURE 1. The principal function  $g(\delta, \gamma)$  of  $T$  where  $v_1 = \alpha v_2 + u$ ,  $\alpha = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ ,  $u \perp v_1$  and  $\|v_2\| = \|u\| = 1$ .

#### REFERENCES

- [1] L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of  $C^*$ -algebras*, Proceedings of a Conference on Operator Theory (Dalhousie Univ., Halifax, N.S., 1973), Springer, Berlin, 1973, pp. 58–128. Lecture Notes in Math., Vol. 345.
- [2] J. B. Conway and P. McGuire, *Operators with  $C^*$ -algebra generated by a unilateral shift*, Trans. Amer. Math. Soc. **284** (1984), no. 1, 153–161.
- [3] R. W. Carey and J. D. Pincus, *An invariant for certain operator algebras*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 1952–1956.
- [4] ———, *Mosaics, principal functions, and mean motion in von Neumann algebras*, Acta Math. **138** (1977), no. 3-4, 153–218.
- [5] ———, *Construction of seminormal operators with prescribed mosaic*, Indiana Univ. Math. J. **23** (1973/74), 1155–1165.
- [6] M. Martin and M. Putinar, *Lectures on hyponormal operators*, Operator Theory: Advances and Applications, vol. 39, Birkhäuser Verlag, Basel, 1989.
- [7] D. Voiculescu, *Free probability for pairs of faces I*, Comm. Math. Phys. **332** (2014), no. 3, 955–980.
- [8] ———, *Free probability for pairs of faces II: 2-variables bi-free partial  $R$ -transform and systems with rank  $\leq 1$  commutation*, preprint, available at [arXiv:1308.2035](https://arxiv.org/abs/1308.2035).

K. DYKEMA, DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843-3368, USA  
*E-mail address*: [kdykema@math.tamu.edu](mailto:kdykema@math.tamu.edu)

W. NA, DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843-3368, USA  
*E-mail address*: [wonhee@math.tamu.edu](mailto:wonhee@math.tamu.edu)